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Chaotic Attitude Tumbling of an Asymmetric Gyrostat in a Gravitational Field

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The chaotic attitude tumbling of an asymmetric gyrostat is investigated in detail. The gyrostat has three symmetrical wheels along the principal axes rotating about a fixed point under the action of either the gravity torques or the gravity-gradient torques. With the use of the Deprit canonical variables, the Euler attitude equations are transformed into Hamiltonian form. This makes the Poincaré–Arnold–Melnikov (PAM) function developed by Holmes and Marsden applicable. The physical parameters triggering the chaotic attitude are established. The analytical results are checked by using the fourth-order Runge–Kutta simulation in terms of the Euler parameters (quaternions). The relationships of the following physical parameters are established: moments of inertia of carriers and wheels, positions of the mass center, kinetic energy and moment of momentum of the torque-free gyrostat, and initial attitude leading to chaotic motion. The results show that the PAM function is a powerful analytical tool for the treatment of the dynamics of nonlinear gyrostat orientations.

Nomenclature

b	=	$\cos^{-1}(L/G)$
$d\psi/dt$, $d\theta/dt$, $d\phi/dt$	=	Euler angular rates
G_p	=	first integral of areas of the torque-free gyrostat
H_0	=	Hamiltonian of the unperturbed system
H_1	=	defined in Eq. (42) for the gyrostat under the action of the gravity torque; defined in Eq. (134) for the gyrostat satellite
h_x, h_y, h_z	=	components (constants) of the vector sum of the relative angular momenta of the wheels with respect to the body-fixed frame
I	=	$\cos^{-1}(H/G)$
I_{xx}, I_{yy}, I_{zz}	=	principal moments of inertia with respect to the body-fixed axes
L, G, H	=	Deprit angular momenta[11]
l, g, h	=	Deprit angles[11]
$\bar{l}, \bar{b}, \bar{L}$	=	homoclinic solutions of the Deprit variables[11] of the unperturbed gyrostat
$M(g_0)$	=	Poincaré–Arnold–Melnikov (see Refs. 5–7) function
M_g	=	product of the gravity acceleration and the total mass of the gyrostat
n	=	orbital circular frequency when orbit is circular, $\sqrt{(\mu/\rho_c^3)}$
$OXYZ$	=	inertial coordinate system for the gyrostat under the action of the gravity torques; local vertical local horizontal coordinate system for the gyrostat satellite
q_x, q_y, q_z, q_0	=	Euler parameters (quaternions)
T_x, T_y, T_z	=	torque components in the body-fixed frame
x_0, y_0, z_0	=	coordinates of the mass center of the gyrostat under the action of the gravity torques in the body-fixed frame $Oxyz$
$\gamma_x, \gamma_y, \gamma_z$	=	directional cosines
ε	=	distance between the mass center and the fixed point of the gyrostat under the action of the gravity torques, r_0; for the gyrostat satellite subjected to the gravity-gradient torques, $3\mu/(2\rho_c^3)$
μ	=	gravitational attraction constant of the Earth
ρ_c	=	distance of the mass center of the gyrostat satellite from the Earth's center
ψ, θ, ϕ	=	Euler angles
$\omega_x, \omega_y, \omega_z$	=	angular velocity components in the body-fixed frame
$\bar{\omega}_x, \bar{\omega}_y, \bar{\omega}_z$	=	homoclinic solutions of angular velocity components

I. Introduction

BECAUSE of the importance of the research and development of spinning satellites and projectiles, the orientation problems of the gyrostat have attracted the attention of many scientists over the last half-century. Most spacecraft use flywheels in the form of momentum wheels and control moment gyros to control attitude. The dynamics and stability of its configuration are naturally of interest.[1] Hughes[2] collected many theoretical and engineering models of rotational gyrostats under torque-free or gravity-gradient toques. Wittenburg[3] investigated the attitude motion of the gyrostat using Wangerin and Volterra's method. Wittenburg obtained the polhode curves on the inertia ellipsoid of the carrier for different wheel speeds. Hall[4] considered the dynamics of gyrostats containing two axisymmetrical rotors and investigated the use of internal torques to enable a spacecraft to escape from a trap state.

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Generally, it is very difficult to obtain analytical solutions for a gyrostat under the action of torques. In this situation, the nonlinear dynamic system theory is a tool to investigate the complex dynamic behaviour for the attitude motion of the gyrostat. The Poincaré–Arnold–Melnikov (PAM) method[5–7] is an analytical technique to detect the transversal intersections of homoclinic (and heteroclinic) orbits. It measures the distance between the stable and unstable manifolds associated with the saddle points in the Poincaré map of the perturbed dynamic system. The PAM function enables one to investigate analytically the chaotic motion in nonlinear dynamic systems. Chen and Leung[8] elaborated some practical chaotic models by using the PAM function. Holmes and Marsden[9] investigated the horseshoes in the perturbed Hamiltonian systems with two degrees of freedom. Holmes and Marsden[10] also studied both the reduction technique of the PAM method and the existence of the Arnold diffusion for the perturbed integrable Hamiltonian systems with three or more degrees of freedom. By using the Kolmogorov–Arnold–Moser (KAM) theory, Holmes and Marsden[10] showed that some of the resulting tori persist under small perturbations. Holmes and Marsden[6] investigated the Smale horseshoes and the Arnold diffusion for nearly integrable Hamiltonian systems using Lie groups. Tong et al.[7] expressed the attitude motion equations of the gyrostat with a single wheel in the gravitational field by the Deprit canonical variables.[11] This in essence laid a foundation for the application of the PAM function. By the use of the PAM method developed by Holmes and Marsden,[6] the chaotic motion in the sense of the Smale horseshoe was detected theoretically and numerically.[7] Tong and Tabarrok[12] studied the global motion of rigid bodies subjected to small perturbation torques, either conservatively or dissipatively. For the conservative case, they used the PAM to determine the parameter regions corresponding to the chaotic attitude dynamics. For the dissipative case, the PAM function formulated by Wiggins and Shaw[13] was introduced to study the chaotic dynamics of the system. Tong and Tabarrok[14] studied the bifurcation of self-excited rigid bodies under small perturbation torques. Based on the work of Wiggins and Shaw,[13] Kuang et al.[15–17] obtained the generalized PAM function applied to the gyrostat under the action of small perturbation torques, in the form of damping torques plus periodic torques. Tong et al.,[7] Tong and Tabarrok,[12,14] and Kuang et al.[15–17] investigated the chaotic attitude dynamics of gyrostat satellite using the Deprit variables.[11] Readers can find other applications of the Deprit variables for the attitude and orbit dynamics of the gyrostat satellite by Elsabaa,[18] Sansaturio and Vigueras,[19] and Cochran.[20]

With the use of the PAM function, several studies were conducted on the chaotic dynamics of the solid satellite based on the fundamental simplification of the attitude equation of the satellite (but without using the Deprit variables[11]).[21–25] In addition, one may refer to the works of Rumyantsev,[26] Pfeiffer,[27] Kuang and Leung,[28] Longuski and Tsiotras,[29] and Longman et al.,[30] on certain cases of the attitude motions of the complex satellite models. These works motivated the authors to propose this paper. According to Rumyantsev,[26] when the motions of the contained liquid are irrotational, the attitude dynamic equations of a rotational liquid-filled body are identical with those of a solid body to which a rotating gyroscope is joined. Consequently, the conclusions of Kuang et al.[15–17] would also hold for the motions of a liquid-filled body containing irrotational liquid under certain appropriate assumptions.

In this paper, the chaotic attitude tumbling of an asymmetric gyrostat with three wheels in the gravitational field is studied. This topic is different from earlier works[16,17] in which the disturbing toques were assumed to be dissipative small perturbation torques. The purpose here is to provide a more fundamental understanding of the chaotic rotational motion of the asymmetric gyrostat under the influence of the gravity torque or the gravity-gradient torques. No dissipation is assumed. In Sec. II, the mathematical formulation using the Deprit variables[11] is outlined. In Sec. III, the PAM function for the gyrostat under the action of the gravity torques is derived. In Sec. IV, the conditions for the existence of transverse homoclinic points for the attitude motion of the gyrostat under the action of the gravity torques are obtained. In Sec. V, the attitude motions of the gyrostat satellite subjected to the gravity-gradient torques are investigated by using the PAM function. In Sec. VI the fourth-order Runge–Kutta integration algorithms (see Ref. 31) are used to simulate a chaotic attractor given. In Sec. VII, the paper is concluded with a few remarks.

II. Hamiltonian Equations in Terms of Deprit's Canonical Variables[11]

For a gyrostat rotating about a fixed point O, the Eulerian equations of the attitude motion are[2]

$$I_{xx}\frac{d\omega_x}{dt} - (I_{yy} - I_{zz})\omega_y\omega_z + \omega_y h_z - \omega_z h_y = T_x \tag{1}$$

$$I_{yy}\frac{d\omega_y}{dt} - (I_{zz} - I_{xx})\omega_z\omega_x + \omega_z h_x - \omega_x h_z = T_y \tag{2}$$

$$I_{zz}\frac{d\omega_z}{dt} - (I_{xx} - I_{yy})\omega_x\omega_y + \omega_x h_y - \omega_y h_x = T_z \tag{3}$$

where the parameters h_x, h_y, and h_z are components of the vector sum of the relative angular momenta of all of the wheels with respect to the body-fixed frame ($Oxyz$) and are assumed to be constants in this paper (Fig. 1). I_{xx}, I_{yy}, and I_{zz} are principal moments of inertia of the gyrostat in the body-fixed frame. T_x, T_y, and T_z are components of the vector of disturbing external torques in the body-fixed frame. The disturbing external torques may be functions of the angular velocities and the angular positions of the body-fixed frame with respect to the inertial coordinate system (Fig. 1). Equations (1–3) describe the attitude motion of the liquid-filled body, for example, a liquid-filled projectile or a liquid-filled satellite model, when the motions of the contained liquid are irrotational[26] and govern the attitude motions of the gyrostat satellite about the mass center to be shown in Sec. V.

Without loss of generality, throughout this paper one assumes that $I_{xx} > I_{yy} > I_{zz}$. The Euler angles ψ, θ, and ϕ are defined as in Fig. 2, where $Oxyz$ is the body-fixed frame coincident with the principal axes of the system. The rotation sequence is $\psi \to \theta \to \phi$. For the complete determination of the attitude motion, three additional differential equations are required. The variables ω_x, ω_y, and ω_z are expressed in Euler angles and Euler angular rates as

$$\omega_x = \frac{d\psi}{dt}\sin\theta\sin\phi + \frac{d\theta}{dt}\cos\phi \tag{4}$$

$$\omega_y = \frac{d\psi}{dt}\sin\theta\cos\phi - \frac{d\theta}{dt}\sin\phi \tag{5}$$

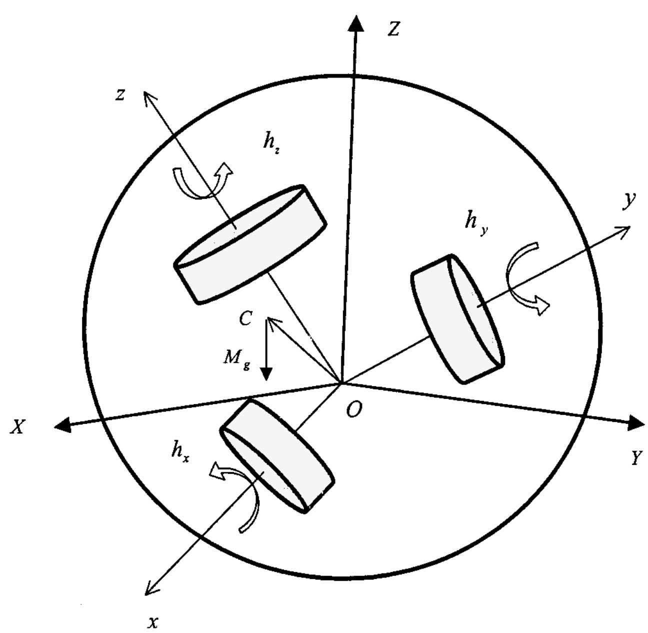


Fig. 1 Configuration of heavy gyrostat rotating about fixed point *O*, where *OC* is the position vector of the mass center *C*, *OXYZ* axes are the inertial coordinate system, and the *Oxyz* are the body-fixed coordinate system.

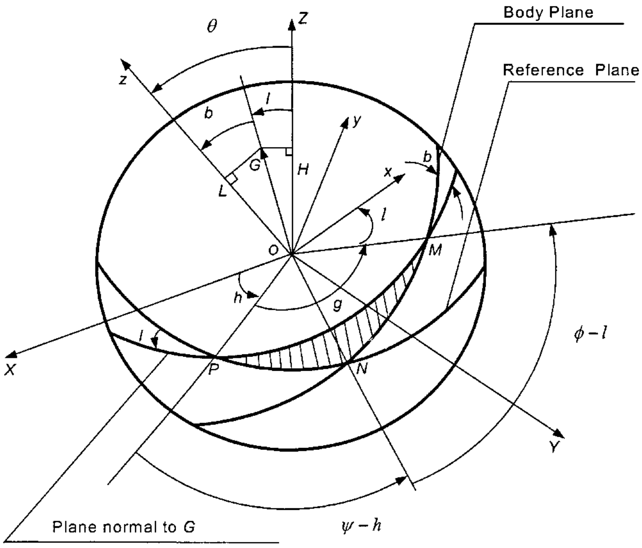


Fig. 2 Euler angles ψ, θ, and ϕ and the Deprit's variables[11] L, G, H, l, g, and h; PNM is a spherical triangle.

$$\omega_z = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt} \tag{6}$$

It is obvious that differential equations (1–6) are nonlinear with singularities with respect to θ. By introducing a set of canonical variables of Deprit,[11] one can establish the Hamiltonian equations of attitude motions that are much more complicated than Eqs. (1–6). However, the new Hamiltonian is ready for the application of the PAM function. The detailed description of the Deprit variables may be found in Ref. 11. Following Deprit, the canonical variables L, G, H, l, g, and h shown in Fig. 2 are introduced. The angles ψ–h, g, and ϕ–l are formulated in terms of the angles I, θ, and b with the usual identities of spherical trigonometry of the spherical triangle PNM (Fig. 2), that is,

$$\cos\theta = \cos I \cos b - \sin I \sin b \cos g \tag{7}$$

$$\sin\theta \cos(\phi - l) = \cos I \sin b + \sin I \cos b \cos g \tag{8}$$

$$\sin\theta \sin(\phi - l) = \sin I \sin g \tag{9}$$

$$\sin\theta \sin(\psi - h) = \sin b \sin g \tag{10}$$

$$\sin\theta \cos(\psi - h) = \sin I \cos b + \cos I \sin b \cos g \tag{11}$$

where variable I is the angle between the momenta H and G and angle b is a function of the momenta G and L, that is,

$$\cos I = H/G \tag{12}$$

$$\cos b = L/G \tag{13}$$

From analytical mechanics, the moment of the momentum of the gyrostat can be expressed in terms of the Deprit canonical variables[11]:

$$I_{xx}\omega_x + h_x = G \sin b \sin l \tag{14}$$

$$I_{yy}\omega_y + h_y = G \sin b \cos l \tag{15}$$

$$I_{zz}\omega_z + h_z = G \cos b \tag{16}$$

By the use of Eqs. (1–16), the Hamiltonian differential equations of the attitude motion of the gyrostat in terms of the Deprit canonical variables[11] were derived by Kuang et al.[15,17]

The total kinetic energy of the gyrostat is

$$H_0 = \tfrac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) \tag{17}$$

Assume that γ_x, γ_y, and γ_z are the directional cosines of the OZ axis with respect to the body-fixed frame; then

$$\gamma_x = \sin\theta \sin\phi = \sin I \sin g \cos l + (\cos I \sin b + \sin I \cos b \cos g)\sin l \tag{18}$$

$$\gamma_y = \sin\theta \cos\phi = -\sin I \sin g \sin l + (\cos I \sin b + \sin I \cos b \cos g)\cos l \tag{19}$$

$$\gamma_z = \cos\theta = \cos I \cos b - \sin I \sin b \cos g \tag{20}$$

Consequently, the Hamiltonian differential equations of the gyrostat in terms of the Deprit variables[11] can be expressed as

$$\frac{dL}{dt} = -\frac{\partial H_0}{\partial l} + \varepsilon f_L(L, G, H, l, g, t) \tag{21}$$

$$\frac{dl}{dt} = \frac{\partial H_0}{\partial L} + \varepsilon f_l(L, G, H, l, g, t) \tag{22}$$

$$\frac{dG}{dt} = \varepsilon f_G(L, G, H, l, g, t) \tag{23}$$

$$\frac{dg}{dt} = \frac{\partial H_0}{\partial G} + \varepsilon f_g(L, G, H, l, g, h, t) \tag{24}$$

$$\frac{dH}{dt} = \varepsilon f_H(L, G, H, l, g, h, t) \tag{25}$$

$$\frac{dh}{dt} = \varepsilon f_h(L, G, H, l, g, h, t) \tag{26}$$

where $0 < \varepsilon = r_0$ is a small parameter and $r_0 = \surd(x_0^2 + y_0^2 + z_0^2)$ is the distance between the fixed point and the mass center of the gyrostat as in Fig. 1. Here, x_0, y_0 and z_0 are the coordinates of the mass center of the gyrostat in the body-fixed frame $Oxyz$. M_g is the product of the total mass of the gyrostat (a carrier plus wheels) and the acceleration of gravity, and H_0 is the kinetic energy function of the gyrostat expressed in Eq. (17). Substituting Eqs. (14–16) into Eq. (17), one obtains

$$H_0 = \tfrac{1}{2}\Big[(G \sin b \sin l - h_x)^2\big/I_{xx} + (G \sin b \cos l - h_y)^2\big/I_{yy} + (G \cos b - h_z)^2\big/I_{zz}\Big] \tag{27}$$

with

$$\frac{\partial H_0}{\partial l} = (G \sin b \cos l)\omega_x - (G \sin b \sin l)\omega_y \tag{28}$$

$$\frac{\partial H_0}{\partial L} = -\frac{\cos b \sin l}{\sin b}\omega_x - \frac{\cos b \cos l}{\sin b}\omega_y + \omega_z \tag{29}$$

$$\frac{\partial H_0}{\partial G} = \frac{\sin l}{\sin b}\omega_x + \frac{\cos l}{\sin b}\omega_y \tag{30}$$

$$f_L = M_z \tag{31}$$

$$f_G = M_z \cos b + (M_x \sin l + M_y \cos l)\sin b \tag{32}$$

$$f_H = M_x[(\sin I \sin g)\cos l + (\cos I \sin b + \sin I \cos b \cos g)\sin l] + M_y[-(\sin I \sin g)\sin l + (\cos I \sin b + \sin I \cos b \cos g)\cos l] + M_z[\cos I \cos b - \sin I \sin b \cos g] = 0 \tag{33}$$

$$f_l = \frac{M_x \cos l - M_y \sin l}{G \sin b} \tag{34}$$

$$f_g = M_x \frac{-(\sin I \cos b + \cos I \sin b \cos g)\cos l + (\cos I \sin b \cos b \sin g)\sin l}{G \sin I \sin b} + M_y \frac{(\sin I \cos b + \cos I \sin b \cos g)\sin l + (\cos I \sin b \cos b \sin g)\cos l}{G \sin I \sin b} + M_z \frac{(-\cos I \sin b \sin g)}{G \sin I} \tag{35}$$

$$f_h = M_x \frac{\cos g \cos l - \cos b \sin g \sin l}{G \sin I} - M_y \frac{\cos g \sin l + \cos b \sin g \cos l}{G \sin I} + M_z \frac{\sin b \sin g}{G \sin I} \tag{36}$$

$$\frac{T_x}{M_x} = \frac{T_y}{M_y} = \frac{T_z}{M_z} = \varepsilon \tag{37}$$

$$M_x = \frac{M_g(\gamma_y z_0 - \gamma_z y_0)}{r_0} \tag{38}$$

$$M_y = \frac{M_g(\gamma_z x_0 - \gamma_x z_0)}{r_0} \tag{39}$$

$$M_z = \frac{M_g(\gamma_x y_0 - \gamma_y x_0)}{r_0} \tag{40}$$

where γ_x, γ_y, and γ_z are defined in terms of the Deprit variables[11] in Eqs. (18–20).

Equations (21) and (22) degenerate into the torque-free motion of the gyrostat when $\varepsilon = 0$. The variable H becomes a constant because the variable h does not appear on the right-hand sides of Eqs. (21–23). Therefore, Eqs. (21–23) are uncoupled from Eqs. (24–26). The following sections will be contributed to determine whether or not the stable and unstable manifolds of the orbit intersect by using the PAM function developed by Holmes and Marsden.[6]

III. Derivation of the PAM Function for the Gyrostat

The Hamiltonian of the gyrostat is

$$H_a = H_0 + \varepsilon H_1 \tag{41}$$

where H_0 is defined in Eq. (17) and

$$H_1 = M_g(x_0\gamma_x + y_0\gamma_y + z_0\gamma_z)/r_0 \tag{42}$$

By the transformations (14–16) and Eqs. (18–20), one can easily express the Hamiltonian in Eq. (41) as a function of the Deprit variables.[11] When $\varepsilon = r_0 = 0$, the attitude problem of the gyrostat reduces to a torque-free motion. Both the momentum $G = G_p$ and the new momentum H are constants. The torque-free motion of the asymmetrical gyrostat may be described by a single-degree-of-freedom Hamiltonian system with the following equations of motion from Eqs. (21–23):

$$\frac{d\bar{L}}{dt} = -\frac{\partial H_0}{\partial \bar{l}} \tag{43}$$

$$\frac{d\bar{l}}{dt} = \frac{\partial H_0}{\partial \bar{L}} \tag{44}$$

where the overbar stands for the solutions to the torque-free gyrostat. Equations (43) and (44) are equivalent to Eqs. (1–3) when $T_x = T_y = T_z = 0$. By using the Wangerin method developed by Wittenburg,[3] one obtains the homoclinic solutions of the angular velocities in Eqs. (1–3) for the torque-free gyrostat:

$$\bar{\omega}_x(t) = \frac{\sum_{n=0}^{12} \omega_{xn} \tanh^n u}{\sum_{n=0}^{12} R_n \tanh^n u} \tag{45}$$

$$\bar{\omega}_y(t) = \frac{\sum_{n=0}^{12} \omega_{yn} \tanh^n u}{\sum_{n=0}^{12} R_n \tanh^n u} \tag{46}$$

$$\bar{\omega}_z(t) = \frac{\sum_{n=0}^{12} \omega_{zn} \tanh^n u}{\sum_{n=0}^{12} R_n \tanh^n u} \tag{47}$$

where $u = \beta(t - t_0)$ and t_0 is the initial time. Other coefficients ω_{xn}, ω_{yn}, and ω_{zn}, R_n, $n = 0, 1, 2, \ldots, 12$, and β can be found in Ref. 16. All of these coefficients are dependent on the following parameters: moments of inertia I_{xx}, I_{yy}, and I_{zz}, the first integrals of energy H_0, and areas G_p of the torque-free gyrostat. The relative equilibrium conditions for the torque-free gyrostat to be hyperbolic can be found by Wittenburg[3] and Kuang et al.[16]

Following Holmes and Marsden,[6] one can detect whether the stable and unstable manifolds of the periodic orbits intersect by calculating the following PAM integral:

$$M(g_0) = \int_{-\infty}^{+\infty} \left\{ H_0, \frac{H_1}{\Omega} \right\} [\bar{l}(t), \bar{L}(t), \bar{g}(t) + g_0]\, dt \tag{48}$$

where g_0 is an integral constant and { } is the Poisson bracket in the variables l and L, that is,

$$\{H_0, (H_1/\Omega)\} = (1/\Omega)\{H_0, H_1\} - (H_1/\Omega^2)\{H_0, \Omega\} \tag{49}$$

with

$$\{H_0, H_1\} = \frac{\partial H_0}{\partial l}\frac{\partial H_1}{\partial L} - \frac{\partial H_0}{\partial L}\frac{\partial H_1}{\partial l} \tag{50}$$

$$\{H_0, \Omega\} = \frac{\partial H_0}{\partial l}\frac{\partial \Omega}{\partial L} - \frac{\partial H_0}{\partial L}\frac{\partial \Omega}{\partial l} \tag{51}$$

$$\Omega = \left.\frac{dg}{dt}\right|_{\varepsilon=0} = \frac{\partial H_0}{\partial G} \tag{52}$$

and $\bar{l}(t)$, $\bar{L}(t)$, and $\bar{g}(t)$ correspond to the homoclinic solutions $\bar{\omega}_x(t)$, $\bar{\omega}_y(t)$, and $\bar{\omega}_z(t)$ in Eqs. (45–47) by the transformations (14–16) and Eq. (52). Using the solutions $\bar{\omega}_x(t)$, $\bar{\omega}_y(t)$, and $\bar{\omega}_z(t)$ instead of $\bar{l}(t)$ and $\bar{L}(t)$, one can make the computation of the PAM integral more efficient. The procedures of dealing with the PAM integral are given in the following section.

IV. Procedures for Dealing with PAM Integral

An explicit evaluation of the PAM integral is quite restricted by the algebraic complexity of the homoclinic solutions involved. By substitution of Eqs. (50–52) together with Eqs. (28–30) and (42) into Eq. (48) and by the use of Eqs. (18–20) together with the transformations (14–16), one can change the PAM integral into the following form:

$$M(g_0) = \int_{-\infty}^{+\infty} [f_0(t, x_0, y_0, z_0) + f_1(t, x_0, y_0, z_0)\cos g + f_2(t, x_0, y_0, z_0)\sin g]\, dt \tag{53}$$

where

$$g(t) = \int_0^t \Omega(s)\, ds + g_0 = \bar{g}(t) + g_0 \tag{54}$$

$$f_k(t, x_0, y_0, z_0) = x_0 \frac{\partial f_k}{\partial x_0}(t) + y_0 \frac{\partial f_k}{\partial y_0}(t) + z_0 \frac{\partial f_k}{\partial z_0}(t) \tag{55}$$

$$\frac{\partial f_k}{\partial x_0}(t) = \frac{1}{\Omega}\left(\frac{\partial H_0}{\partial l}\frac{\partial \gamma_{xk}}{\partial L} - \frac{\partial H_0}{\partial L}\frac{\partial \gamma_{xk}}{\partial l}\right) - \frac{\gamma_{xk}}{\Omega^2}\left(\frac{\partial H_0}{\partial l}\frac{\partial \Omega}{\partial L} - \frac{\partial H_0}{\partial L}\frac{\partial \Omega}{\partial l}\right) \tag{56}$$

$$\frac{\partial f_k}{\partial y_0}(t) = \frac{1}{\Omega}\left(\frac{\partial H_0}{\partial l}\frac{\partial \gamma_{yk}}{\partial L} - \frac{\partial H_0}{\partial L}\frac{\partial \gamma_{yk}}{\partial l}\right) - \frac{\gamma_{yk}}{\Omega^2}\left(\frac{\partial H_0}{\partial l}\frac{\partial \Omega}{\partial L} - \frac{\partial H_0}{\partial L}\frac{\partial \Omega}{\partial l}\right) \tag{57}$$

$$\frac{\partial f_k}{\partial z_0}(t) = \frac{1}{\Omega}\left(\frac{\partial H_0}{\partial l}\frac{\partial \gamma_{zk}}{\partial L} - \frac{\partial H_0}{\partial L}\frac{\partial \gamma_{zk}}{\partial l}\right) - \frac{\gamma_{zk}}{\Omega^2}\left(\frac{\partial H_0}{\partial l}\frac{\partial \Omega}{\partial L} - \frac{\partial H_0}{\partial L}\frac{\partial \Omega}{\partial l}\right) \tag{58}$$

where $k = 0, 1, 2$ and

$$\Omega = \frac{G_p\lfloor(I_{xx}\bar{\omega}_x + h_x)\bar{\omega}_x + (I_{yy}\bar{\omega}_y + h_y)\bar{\omega}_y\rfloor}{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2} \tag{59}$$

$$\frac{\partial H_0}{\partial l} = (I_{yy}\bar{\omega}_y + h_y)\bar{\omega}_x - (I_{xx}\bar{\omega}_x + h_x)\bar{\omega}_y \tag{60}$$

$$\frac{\partial H_0}{\partial L} = \bar{\omega}_z - \frac{(I_{zz}\bar{\omega}_z + h_z)\lfloor(I_{xx}\bar{\omega}_x + h_x)\bar{\omega}_x + (I_{yy}\bar{\omega}_y + h_y)\bar{\omega}_y\rfloor}{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2} \tag{61}$$

$$\frac{\partial \Omega}{\partial l} = \frac{G_p\lfloor(I_{yy}\bar{\omega}_y + h_y)[2(I_{yy} - I_{xx})(I_{xx}\bar{\omega}_x + h_x) - I_{yy}h_x] + I_{xx}(I_{xx}\bar{\omega}_x + h_x)h_y\rfloor}{I_{xx}I_{yy}\left[(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2\right]} \tag{62}$$

$$\frac{\partial \Omega}{\partial L} = -\left[\frac{h_x(I_{xx}\bar{\omega}_x + h_x)}{I_{xx}} + \frac{h_y(I_{yy}\bar{\omega}_y + h_y)}{I_{yy}}\right] \times \frac{G_p(I_{zz}\bar{\omega}_z + h_z)}{\left[(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2\right]^2} \tag{63}$$

$$\gamma_{k0} = \frac{\cos I\,(I_{kk}\bar{\omega}_k + h_k)}{G_p} \tag{64}$$

where $k = x, y, z$ and

$$\gamma_{x1} = \frac{\sin I\,(I_{zz}\bar{\omega}_z + h_z)(I_{xx}\bar{\omega}_x + h_x)}{\pm G_p\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{65}$$

$$\gamma_{y1} = \frac{\sin I\,(I_{zz}\bar{\omega}_z + h_z)(I_{yy}\bar{\omega}_y + h_y)}{\pm G_p\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{66}$$

$$\gamma_{z1} = \frac{-\sin I\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}}{\pm G_p} \tag{67}$$

$$\gamma_{x2} = \frac{\sin I\,(I_{yy}\bar{\omega}_y + h_y)}{\pm\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{68}$$

$$\gamma_{y2} = \frac{-\sin I\,(I_{xx}\bar{\omega}_x + h_x)}{\pm\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{69}$$

$$\gamma_{z2} = \frac{\partial \gamma_{z2}}{\partial l} = \frac{\partial \gamma_{z2}}{\partial L} = 0 \tag{70}$$

$$\frac{\partial \gamma_{x0}}{\partial l} = \frac{\cos I\,(I_{yy}\bar{\omega}_y + h_y)}{G_p} \tag{71}$$

$$\frac{\partial \gamma_{y0}}{\partial l} = \frac{-\cos I\,(I_{xx}\bar{\omega}_x + h_x)}{G_p} \tag{72}$$

$$\frac{\partial \gamma_{z0}}{\partial l} = \frac{\partial \gamma_{z1}}{\partial l} = 0 \tag{73}$$

$$\frac{\partial \gamma_{x1}}{\partial l} = \frac{\sin I\,(I_{zz}\bar{\omega}_z + h_z)(I_{yy}\bar{\omega}_y + h_y)}{\pm G_p\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{74}$$

$$\frac{\partial \gamma_{y1}}{\partial l} = \frac{-\sin I\,(I_{zz}\bar{\omega}_z + h_z)(I_{xx}\bar{\omega}_x + h_x)}{\pm G_p\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{75}$$

$$\frac{\partial \gamma_{x2}}{\partial l} = \frac{-\sin I\,(I_{xx}\bar{\omega}_x + h_x)}{\pm\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{76}$$

$$\frac{\partial \gamma_{y2}}{\partial l} = \frac{-\sin I\,(I_{yy}\bar{\omega}_y + h_y)}{\pm\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{77}$$

$$\frac{\partial \gamma_{x0}}{\partial L} = \frac{-\cos I\,(I_{zz}\bar{\omega}_z + h_z)(I_{xx}\bar{\omega}_x + h_x)}{\pm G_p\left[(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2\right]} \tag{78}$$

$$\frac{\partial \gamma_{y0}}{\partial L} = \frac{-\cos I\,(I_{zz}\bar{\omega}_z + h_z)(I_{yy}\bar{\omega}_y + h_y)}{\pm G_p\left[(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2\right]} \tag{79}$$

$$\frac{\partial \gamma_{z0}}{\partial L} = \frac{\cos I}{G_p} \tag{80}$$

$$\frac{\partial \gamma_{x1}}{\partial L} = \frac{\sin I\,(I_{xx}\bar{\omega}_x + h_x)}{\pm G_p\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{81}$$

$$\frac{\partial \gamma_{y1}}{\partial L} = \frac{\sin I\,(I_{yy}\bar{\omega}_y + h_y)}{\pm G_p\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{82}$$

$$\frac{\partial \gamma_{z1}}{\partial L} = \frac{\sin I\,(I_{zz}\bar{\omega}_z + h_z)}{\pm G_p\sqrt{(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2}} \tag{83}$$

$$\frac{\partial \gamma_{x2}}{\partial L} = \frac{\partial \gamma_{y2}}{\partial L} = 0 \tag{84}$$

By introducing the transformation

$$z = \tanh[\beta(t - t_0)] \tag{85}$$

and setting

$$F_k(z, x_0, y_0, z_0) = f_k(t, x_0, y_0, z_0) \tag{86}$$

where $k = 0, 1, 2$ and

$$\bar{G}(z) = \bar{g}(t) \tag{87}$$

from Eq. (53), one obtains

$$M(g_0) = I_0 + I_c\cos(g_0) + I_s\sin(g_0) \tag{88}$$

where

$$I_0 = \int_{-1}^{+1}\frac{F_0(z, x_0, y_0, z_0)}{\beta(1 - z^2)}\,\mathrm{d}z \tag{89}$$

$$I_c = \int_{-1}^{+1}\frac{F_1(z, x_0, y_0, z_0)\cos[\bar{G}(z)] + F_2(z, x_0, y_0, z_0)\sin[\bar{G}(z)]}{\beta(1 - z^2)}\,\mathrm{d}z \tag{90}$$

$$I_s = \int_{-1}^{+1} \frac{-F_1(z, x_0, y_0, z_0)\sin[\bar{G}(z)] + F_2(z, x_0, y_0, z_0)\cos[\bar{G}(z)]}{\beta(1-z^2)}\,dz \tag{91}$$

From Eqs. (64–84), one finds that the integral I_0 in Eq. (89) contains a factor $\cos I$ and that the integrals I_c in Eq. (90) and I_s in Eq. (91) contain a factor $\sin I$. To eliminate the singularities (at $z = \pm 1$) in the integrals in Eqs. (89–91), it is required that

$$F_i(\pm 1, x_0, y_0, z_0) = 0 \qquad (i = 0, 1, 2) \tag{92}$$

According to the properties of the homoclinic orbits of Eqs. (43) and (44) for the torque-free gyrostat, one finds that

$$\left.\frac{\partial H_0}{\partial l}\right|_{t\to\pm\infty} = 0 \tag{93}$$

$$\left.\frac{\partial H_0}{\partial L}\right|_{t\to\pm\infty} = 0 \tag{94}$$

along the homoclinic solutions in Eqs. (45–47). Consequently, one can say that Eq. (92) holds always. If one assumes that

$$\left|I_0\big/\sqrt{I_s^2 + I_c^2}\right| \le 1 \tag{95}$$

then

$$M(g_0) = I_0 + \sqrt{I_s^2 + I_c^2}\sin\left[g_0 + \tan^{-1}(I_c/I_s)\right] \tag{96}$$

will have simple zeros with respect to the variable g_0. The constraint equation (95) ensures that $M(g_0)$ changes sign for some g_0. If the inequality (95) did not hold, then the PAM function in Eq. (96) would not have simple zeros with respect to the variable g_0. One concludes that the PAM function in Eq. (96) has simple zeros if Eq. (95) holds. This result shows that there exist transversely intersections between the stable and unstable manifolds of the gyrostat motion under the gravity torques.

From Eqs. (64–91), one observes that, if

$$(I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2 = 0 \tag{97}$$

or

$$(I_{xx}\bar{\omega}_x + h_x)\bar{\omega}_x + (I_{yy}\bar{\omega}_y + h_y)\bar{\omega}_y = 0 \tag{98}$$

singularities will also exist in the integrals in Eqs. (89–91). Fortunately, according to the first integrals of energy and areas of the torque-free gyrostat, one can prove that Eq. (97) does not hold when one carefully chooses the kinetic energy constant H_0 for the torque-free gyrostat (see subsequent Remark). Substituting Eqs. (45–47) and (85) into Eq. (98), one obtains

$$\frac{\sum_{n=0}^{24}\Gamma_n z^n}{\left(\sum_{n=0}^{12} R_n z^n\right)^2} = 0 \tag{99}$$

where

$$\Gamma_n = \Gamma_{nx} + \Gamma_{ny} \tag{100}$$

$$\Gamma_{nk} = \sum_{p=0}^{n}\theta_{k,n-p}\omega_{kp} \qquad (0 \le n \le 12,\ k = x, y) \tag{101}$$

$$\Gamma_{nk} = \sum_{p=0}^{24-n}\theta_{k,12-p}\omega_{k,p+n-12} \qquad (13 \le n \le 24,\ k = x, y) \tag{102}$$

$$\theta_{xn} = \omega_{xn} + h_x R_n \tag{103}$$

$$\theta_{yn} = \omega_{yn} + h_y R_n \tag{104}$$

The analytical method developed by Wittenburg[3] and Kuang et al.[16] shows that the denominator on the left side in Eq. (99) does not equal to zero.[16] Equation (99) becomes

$$\sum_{n=0}^{24}\Gamma_n z^n = 0 \tag{105}$$

The singularities of the PAM integral in $z \in (-1, +1)$ depend on whether Eq. (105) has real zeros in the range. The following remark provides an alternative method for further details.

Remark: The discussion for the singularities of the PAM function in the application of the attitude motion of the gyrostat under the gravity torques is given as follows. The contradiction argument is used for reasoning. For the torque-free gyrostat, the first integral of the kinetic energy is expressed as

$$2H_0 = I_{xx}\bar{\omega}_x^2 + I_{yy}\bar{\omega}_y^2 + I_{zz}\bar{\omega}_z^2 = \text{const} \tag{106}$$

and the first integral of areas is

$$G_p^2 = (I_{xx}\bar{\omega}_x + h_x)^2 + (I_{yy}\bar{\omega}_y + h_y)^2 + (I_{zz}\bar{\omega}_z + h_z)^2 = \text{const} \tag{107}$$

From the transformations (14) and (15) between the Euler angular velocities and the Deprit's variables[11] for the torque-free gyrostat, the following equations hold:

$$I_{xx}\bar{\omega}_x + h_x = G_p \sin\bar{b}\sin\bar{l} \tag{108}$$

$$I_{yy}\bar{\omega}_y + h_y = G_p \sin\bar{b}\cos\bar{l} \tag{109}$$

$$I_{zz}\bar{\omega}_z + h_z = G_p \cos\bar{b} \tag{110}$$

where the overbar stands for the homoclinic solutions with respect to the attitude motion of the torque-free gyrostat. Assuming that Eq. (97) holds, then one obtains

$$I_{xx}\bar{\omega}_x + h_x = I_{yy}\bar{\omega}_y + h_y = 0 \tag{111}$$

Substitution of Eqs. (111) into Eqs. (106) and (107) leads to the result

$$H_0 = \tfrac{1}{2}\left[h_x^2\big/I_{xx} + h_y^2\big/I_{yy} + (G_p \pm h_z)^2\big/I_{zz}\right] \tag{112}$$

An alternative to Eq. (105) can be exploited to check the nonexistence of the singularities by means of relation (98). Assume that relation (112) does not hold, then from Eqs. (108–111) one obtains the following inequalities:

$$\sin\bar{b} \ne 0, \qquad \sin\bar{l} \ne 0, \qquad \cos\bar{l} \ne 0 \tag{113}$$

Further, one assumes that relation (98) holds. Substituting Eqs. (108–110) into Eq. (106) and making use of relation (113), one obtains the following equations:

$$\sum_{i=0}^{8} b_i \tan^i(\bar{l}) = 0 \tag{114}$$

where

$$b_8 = \Omega_4^2 + h_x^2 \tag{115}$$

$$b_7 = 2\Omega_4\Omega_3 + 2h_x h_y \tag{116}$$

$$b_6 = 2\Omega_4\Omega_2 + \Omega_3^2 + h_y^2 + 3h_x^2 \tag{117}$$

$$b_5 = 2\Omega_2\Omega_3 + 2\Omega_1\Omega_4 + 6h_x h_y \tag{118}$$

$$b_4 = 2\Omega_1\Omega_3 + \Omega_2^2 + 2\Omega_0\Omega_4 + 3h_y^2 + 3h_x^2 \tag{119}$$

$$b_3 = 2\Omega_1\Omega_2 + 2\Omega_0\Omega_3 + 6h_x h_y \tag{120}$$

$$b_2 = 2\Omega_0\Omega_2 + \Omega_1^2 + h_x^2 + 3h_y^2 \tag{121}$$

$$b_1 = 2\Omega_0\Omega_1 + 2h_x h_y \tag{122}$$

$$b_0 = \Omega_0^2 + h_y^2 \tag{123}$$

$$\Omega_4 = \frac{I_{yy}\left(G_p^2 + h_z^2 - h_x^2\right) + I_{zz}h_y^2 - 2I_{yy}I_{zz}H_0}{2I_{yy}h_z} \tag{124}$$

$$\Omega_3 = \frac{-(I_{yy} + I_{zz})h_x h_y}{I_{yy}h_z} \tag{125}$$

$$\Omega_2 = \frac{(I_{zz} - I_{yy})I_{xx}h_x^2 + (I_{zz} - I_{xx})I_{yy}h_y^2 + 2I_{xx}I_{yy}\left(G_p^2 + h_z^2\right) - 4I_{xx}I_{yy}H_0}{2I_{xx}I_{yy}h_z} \tag{126}$$

$$\Omega_1 = \frac{-(I_{xx} + I_{zz})h_x h_y}{I_{xx}h_z} \tag{127}$$

$$\Omega_0 = \frac{I_{xx}\left(G_p^2 + h_z^2 - h_y^2\right) + I_{zz}h_x^2 - 2I_{xx}I_{zz}H_0}{2I_{xx}h_z} \tag{128}$$

The existence of singularities represented by Eq. (98) can be judged by finding the possible real roots of the eight-degree polynomial equation in $\tan(\bar{l})$ in Eq. (114). If all of the roots of the polynomial Eq. (114) are complex, then there are no singularities in the PAM function. If Eq. (105) in the range $(-1, +1)$ or Eq. (114) has simple real zeros, the singularities in the integral will make the PAM method not applicable to the gyrostat system directly. When $M(g_0)$ in Eq. (96) has simple zeros with respect to the g_0, one can conclude that there exist transversely homoclinic orbits for the attitude motion of the gyrostat. Thus, the motion of the heavy gyrostat rotating about a fixed point is chaotic in the sense of Smale's horseshoes.

V. Chaotic Motions of a Gyrostat Satellite Subject to the Gravity-Gradient Torques

Gravitational torques are fundamental to the study of attitude dynamics of the satellite. If the gravitational field were uniform over the satellite, then the center of the satellite mass would have become the center of gravity, and the gravitational torques about the mass center of the satellite would have been zero. In outer space, the gravitational field is not uniform. The gravitational effects were extensively considered in celestial mechanics papers.[2,21] In this section, artificial satellites in the gravitational field of the Earth are investigated. Assume that a gyrostat in a circular orbit possesses a body-fixed orthogonal frame *Oxyz* that is coincident with the principal central axes of moments of inertia. The local reference orthogonal frame with its origin at the center of mass of the gyrostat satellite is *OXYZ*, with the axis *OX* perpendicular to the orbit plane, *OY* along the orbit direction, and *OZ* outward the Earth center so that *OXYZ* forms a right-hand frame, in Fig. 3. The origin *O* is assumed to coincide with the mass center of the gyrostat. The three wheels are rotating with constant relative angular momenta h_x, h_y, and h_z, respectively, of the gyrostat satellite with respect to the body-fixed frame *Oxyz* (Fig. 3). According to Hughes,[2] under appropriate assumptions, the gravity-gradient torques with respect to the body-fixed frame are written as

$$T_x = \left(3\mu/\rho_c^3\right)(I_{zz} - I_{yy})\gamma_z\gamma_y \tag{129}$$

$$T_y = \left(3\mu/\rho_c^3\right)(I_{xx} - I_{zz})\gamma_x\gamma_z \tag{130}$$

$$T_z = \left(3\mu/\rho_c^3\right)(I_{yy} - I_{xx})\gamma_x\gamma_y \tag{131}$$

where μ is the gravitational attraction constant of the Earth and ρ_c is the distance of the mass center of the gyrostat satellite from the Earth center. I_{xx}, I_{yy}, and I_{zz} are the centroidal principal moments of inertia of the gyrostat satellite in the body-fixed frame (*Oxyz*). Equations (129–131) represent the first terms of a Taylor series approximation of the exact gravity moment equations associated with a body in an inverse square gravitational field. It is known that, if two centroidal principal moments of inertia are equal, the gravity-gradient torque about the third axis is always equal to zero. For a tri-inertial gyrostat, that is, I_{xx}, I_{yy}, and I_{zz} are unequal, it is obvious that the gravity-gradient torques vanish if and only if two of the three direction cosines are zeros. In geometric terms, one of the principal axes must be aligned with the local vertical. The infinitesimal static stability of these equilibria is considered by Hughes.[2] When one ignores the constant part of the Keplerian potential, the expression for the gravitational potential can be interpreted as

$$V = \left(3\mu/2\rho_c^3\right)\left(I_{xx}\gamma_x^2 + I_{yy}\gamma_y^2 + I_{zz}\gamma_z^2\right) - \left(\mu/2\rho_c^3\right)(I_{xx} + I_{yy} + I_{zz}) = \varepsilon H_1 \tag{132}$$

where

$$\varepsilon = 3\mu/2\rho_c^3 \tag{133}$$

$$H_1 = \left(I_{xx}\gamma_x^2 + I_{yy}\gamma_y^2 + I_{zz}\gamma_z^2\right) - \tfrac{1}{3}(I_{xx} + I_{yy} + I_{zz}) \tag{134}$$

in which γ_x, γ_y, and γ_z are defined in Eqs. (18–20). The disturbance torque in the body-fixed frame is due to the gravity-gradient torques. The differential equations of the attitude motion are similar to Eqs. (1–6) in the classic form or Eqs. (21–26) in terms of the Deprit variables.[11]

From Eq. (134), one derives the following formulas:

$$\frac{\partial H_1}{\partial l} = 2\left(I_{xx}\gamma_x\frac{\partial\gamma_x}{\partial l} + I_{yy}\gamma_y\frac{\partial\gamma_y}{\partial l} + I_{zz}\gamma_z\frac{\partial\gamma_z}{\partial l}\right) \tag{135}$$

$$\frac{\partial H_1}{\partial L} = 2\left(I_{xx}\gamma_x\frac{\partial\gamma_x}{\partial L} + I_{yy}\gamma_y\frac{\partial\gamma_y}{\partial L} + I_{zz}\gamma_z\frac{\partial\gamma_z}{\partial L}\right) \tag{136}$$

Substituting Eqs. (18–20) and (64–84) into Eqs. (134–136), one obtains

$$H_1 = k_0(t) + k_1(t)\cos(g) + k_2(t)\sin(g) + k_3(t)\sin(2g) + k_4(t)\cos(2g) \tag{137}$$

$$\frac{\partial H_1}{\partial l} = \alpha_0(t) + \alpha_1(t)\cos(g) + \alpha_2(t)\sin(g) + \alpha_3(t)\sin(2g) + \alpha_4(t)\cos(2g) \tag{138}$$

$$\frac{\partial H_1}{\partial L} = \beta_0(t) + \beta_1(t)\cos(g) + \beta_2(t)\sin(g) + \beta_3(t)\sin(2g) + \beta_4(t)\cos(2g) \tag{139}$$

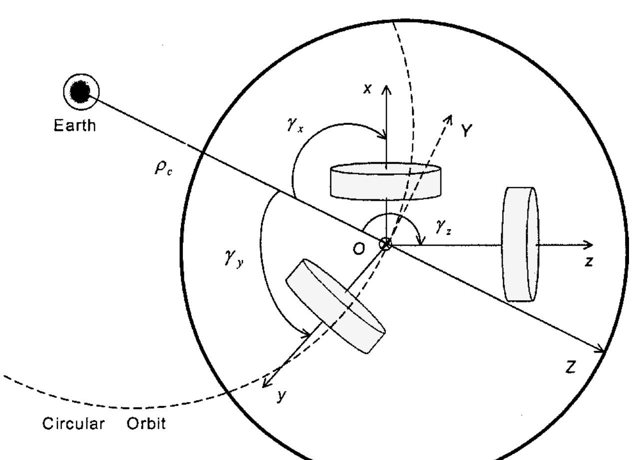


Fig. 3 Configuration of the gyrostat rotating about the center of mass *O* that orbits the Earth in a circular orbit with a radius ρ_c; ⊗ indicates the mass center of the system and the *OX* axis.

where $k_i(t)$, $\alpha_i(t)$, $\beta_i(t)$, $i = 0, 1, 2, 3, 4$, are defined as

$$k_0(t) = \sum_{i=x,y,z} I_{ii}\left(\gamma_{i0}^2 + \frac{1}{2}\gamma_{i1}^2 + \frac{1}{2}\gamma_{i2}^2\right) - \frac{1}{3}(I_{xx} + I_{yy} + I_{zz}) \tag{140}$$

$$k_1(t) = 2\sum_{i=x,y,z} I_{ii}\gamma_{i0}\gamma_{i1} \tag{141}$$

$$k_2(t) = 2\sum_{i=x,y,z} I_{ii}\gamma_{i0}\gamma_{i2} \tag{142}$$

$$k_3(t) = \sum_{i=x,y,z} I_{ii}\gamma_{i1}\gamma_{i2} \tag{143}$$

$$k_4(t) = \frac{1}{2}\sum_{i=x,y,z} I_{ii}\left(\gamma_{i1}^2 - \gamma_{i2}^2\right) \tag{144}$$

$$\alpha_0(t) = 2\sum_{i=x,y,z} I_{ii}\left(\gamma_{i0}\frac{\partial\gamma_{i0}}{\partial l} + \frac{1}{2}\gamma_{i1}\frac{\partial\gamma_{i1}}{\partial l} + \frac{1}{2}\gamma_{i2}\frac{\partial\gamma_{i2}}{\partial l}\right) \tag{145}$$

$$\alpha_1(t) = 2\sum_{i=x,y,z} I_{ii}\left(\gamma_{i1}\frac{\partial\gamma_{i0}}{\partial l} + \gamma_{i0}\frac{\partial\gamma_{i1}}{\partial l}\right) \tag{146}$$

$$\alpha_2(t) = 2\sum_{i=x,y,z} I_{ii}\left(\gamma_{i2}\frac{\partial\gamma_{i0}}{\partial l} + \gamma_{i0}\frac{\partial\gamma_{i2}}{\partial l}\right) \tag{147}$$

$$\alpha_3(t) = \sum_{i=x,y,z} I_{ii}\left(\gamma_{i1}\frac{\partial\gamma_{i2}}{\partial l} + \gamma_{i2}\frac{\partial\gamma_{i1}}{\partial l}\right) \tag{148}$$

$$\alpha_4(t) = \sum_{i=x,y,z} I_{ii}\left(\gamma_{i1}\frac{\partial\gamma_{i1}}{\partial l} - \gamma_{i2}\frac{\partial\gamma_{i2}}{\partial l}\right) \tag{149}$$

$$\beta_0(t) = 2\sum_{i=x,y,z} I_{ii}\left(\gamma_{i0}\frac{\partial\gamma_{i0}}{\partial L} + \frac{1}{2}\gamma_{i1}\frac{\partial\gamma_{i1}}{\partial L} + \frac{1}{2}\gamma_{i2}\frac{\partial\gamma_{i2}}{\partial L}\right) \tag{150}$$

$$\beta_1(t) = 2\sum_{i=x,y,z} I_{ii}\left(\gamma_{i1}\frac{\partial\gamma_{i0}}{\partial L} + \gamma_{i0}\frac{\partial\gamma_{i1}}{\partial L}\right) \tag{151}$$

$$\beta_2(t) = 2\sum_{i=x,y,z} I_{ii}\left(\gamma_{i2}\frac{\partial\gamma_{i0}}{\partial L} + \gamma_{i0}\frac{\partial\gamma_{i2}}{\partial L}\right) \tag{152}$$

$$\beta_3(t) = \sum_{i=x,y,z} I_{ii}\left(\gamma_{i1}\frac{\partial\gamma_{i2}}{\partial L} + \gamma_{i2}\frac{\partial\gamma_{i1}}{\partial L}\right) \tag{153}$$

$$\beta_4(t) = \sum_{i=x,y,z} I_{ii}\left(\gamma_{i1}\frac{\partial\gamma_{i1}}{\partial L} - \gamma_{i2}\frac{\partial\gamma_{i2}}{\partial L}\right) \tag{154}$$

The expression for the PAM integral in Eq. (48) for the attitude motion of the gyrostat satellite under the action of the gravity-gradient torques can be interpreted as

$$M(g_0) = \int_{-\infty}^{+\infty} [\theta_0(t) + \theta_1(t)\cos(g) + \theta_2(t)\sin(g) + \theta_3(t)\sin(2g) + \theta_4(t)\cos(2g)]\,\mathrm{d}t \tag{155}$$

where

$$\theta_i(t) = \frac{1}{\Omega}\left[\frac{\partial H_0}{\partial l}\beta_i(t) - \frac{\partial H_0}{\partial L}\alpha_i(t)\right] - \frac{k_i(t)}{\Omega^2}\left[\frac{\partial H_0}{\partial l}\frac{\partial\Omega}{\partial L} - \frac{\partial H_0}{\partial L}\frac{\partial\Omega}{\partial l}\right] \tag{156}$$

with $i = 0, 1, 2, 3$, and 4.

By using Eqs. (85) and (87) and setting

$$\Theta_i(z) = \theta_i(t), \qquad (i = 0, 1, 2, 3, 4) \tag{157}$$

$$J_0 = \int_{-1}^{+1} \frac{\Theta_0(z)}{\beta(1 - z^2)}\,\mathrm{d}z \tag{158}$$

$$J_1 = \int_{-1}^{+1} \frac{\{\Theta_1(z)\cos[\bar{G}(z)] + \Theta_2(z)\sin[\bar{G}(z)]\}}{\beta(1 - z^2)}\,\mathrm{d}z \tag{159}$$

$$J_2 = \int_{-1}^{+1} \frac{\{-\Theta_1(z)\sin[\bar{G}(z)] + \Theta_2(z)\cos[\bar{G}(z)]\}}{\beta(1 - z^2)}\,\mathrm{d}z \tag{160}$$

$$J_3 = \int_{-1}^{+1} \frac{2\{\Theta_3(z)\cos[2\bar{G}(z)] - \Theta_4(z)\sin[2\bar{G}(z)]\}}{\beta(1 - z^2)}\,\mathrm{d}z \tag{161}$$

$$J_4 = \int_{-1}^{+1} \frac{\{\Theta_3(z)\sin[2\bar{G}(z)] + \Theta_4(z)\cos[2\bar{G}(z)]\}}{\beta(1 - z^2)}\,\mathrm{d}z \tag{162}$$

$$J_5 = -\int_{-1}^{+1} \frac{\{\Theta_3(z)\sin[2\bar{G}(z)] + \Theta_4(z)\cos[2\bar{G}(z)]\}}{\beta(1 - z^2)}\,\mathrm{d}z \tag{163}$$

one can transform Eq. (155) into the following form:

$$M(g_0) = J_0 + J_1\cos(g_0) + J_2\sin(g_0) + J_3\sin(g_0)\cos(g_0) + J_4\cos^2(g_0) + J_5\sin^2(g_0) \tag{164}$$

The singularities at $z = \pm 1$ in integrals (158–163) can be eliminated by using Eqs. (93) and (94) based on the properties of the homoclinic solutions of the torque-free gyrostats. The PAM integrals can be transformed into the following trigonometric form:

$$M(g_0) = \frac{Q_4\tan^4(0.5g_0) + Q_3\tan^3(0.5g_0) + Q_2\tan^2(0.5g_0) + Q_1\tan(0.5g_0) + Q_0}{\tan^4(0.5g_0) + 2\tan^2(0.5g_0) + 1} \tag{165}$$

where

$$Q_4 = J_0 + J_4 - J_1 \tag{166}$$

$$Q_3 = 2(J_2 - J_3) \tag{167}$$

$$Q_2 = 2(J_0 - J_4 + 2J_5) \tag{168}$$

$$Q_1 = 2(J_2 + J_3) \tag{169}$$

$$Q_0 = J_0 + J_4 + J_1 \tag{170}$$

If there exists a real angle g_0 making the PAM function have a simple zero, that is,

$$M(g_0) = 0 \tag{171}$$

where $M(g_0)$ is defined in Eq. (165), then there exist transversal intersections between the stable and unstable manifolds in the attitude motions of the gyrostat satellite under the action of the gravity-gradient torques. The conditions for the existence of real zeros in Eq. (171) are identical with those of the four-degree Eq. (165). The coefficients of Eq. (171) associated with Eqs. (166–170) are complicated functions of the physical parameters of the gyrostat satellite: I_{xx}, I_{yy}, I_{zz}, h_x, h_y, h_z, G_p, and H_0. In addition,

$$J_0 = \frac{\partial J_0}{\partial(\sin^2 I)}\sin^2 I + \frac{\partial J_0}{\partial(\cos^2 I)}\cos^2 I \tag{172}$$

$$J_1 = \frac{\partial J_1}{\partial(\sin I\cos I)}\sin I\cos I \tag{173}$$

$$J_2 = \frac{\partial J_2}{\partial(\sin I\cos I)}\sin I\cos I \tag{174}$$

$$J_3 = \frac{\partial J_3}{\partial(\sin^2 I)}\sin^2 I \tag{175}$$

$$J_4 = \frac{\partial J_4}{\partial(\sin^2 I)}\sin^2 I \tag{176}$$

$$J_5 = \frac{\partial J_5}{\partial(\sin^2 I)}\sin^2 I \tag{177}$$

The partial derivatives in Eqs. (172–177) can be derived from Eqs. (158–163) associated with Eqs. (64–84), (137–154), (156), and (157) by using the MATLAB® software (Version 6.0.0.88, Release 12). Only multiplication and addition both numerically and symbolically are involved because J_k, $k=0,1,\ldots,5$, are linear functions of the $\sin^2 I$, $\cos^2 I$, and $\sin I\cos I$. They are omitted here. From Eqs. (171) and (164), the following equations can be derived:

$$\gamma_{ss}\sin^2 I + \gamma_{sc}\sin I\cos I + \gamma_{cc}\cos^2 I = 0 \tag{178}$$

$$\gamma_{ss} = \frac{\partial J_0}{\partial(\sin^2 I)} + \frac{\partial J_3}{\partial(\sin^2 I)}\sin(g_0)\cos(g_0) + \frac{\partial J_4}{\partial(\sin^2 I)}\cos^2(g_0) + \frac{\partial J_5}{\partial(\sin^2 I)}\sin^2(g_0) \tag{179}$$

$$\gamma_{sc} = \frac{\partial J_1}{\partial(\sin I\cos I)}\cos(g_0) + \frac{\partial J_2}{\partial(\sin I\cos I)}\sin(g_0) \tag{180}$$

$$\gamma_{cc} = \frac{\partial J_0}{\partial(\cos^2 I)} \tag{181}$$

By using trigonometric function formulas, one obtains, from Eq. (178),

$$\gamma_{ss}\tan^2 I + \gamma_{sc}\tan I + \gamma_{cc} = 0 \tag{182}$$

The condition for $\tan I$ to be real is

$$\gamma_{sc}^2 - 4\gamma_{cc}\gamma_{ss} \geq 0 \tag{183}$$

Equation (182) establishes the relationship between the Deprit[11] angle parameter I and g_0 for the torque-free gyrostat satellite whose attitude dynamics will be chaotic when it is subject to the gravity-gradient torques. Inequality (183) ensures that the PAM function in Eq. (165) has simple zeros with respect to the parameter g_0. Some numerical examples will be given in the next section.

VI. Simulation Examples

Two practical models will be discussed. The first one is the chaotic attitude dynamics of a heavy gyrostat rotating about a fixed point under the action of gravity torques. The second one is the chaotic attitude dynamics of a gyrostat satellite under the action of the gravity-gradient torques. The total moments of the principal inertia of the gyrostat are $I_{xx}=12$, $I_{yy}=9.5$, and $I_{zz}=6$ kg $\cdot$ m^2 in the body-fixed frame. The constant relative angular momenta of the wheels are $h_x=5.4768$, $h_y=1.1789$, and $h_z=-13.4327$ N $\cdot$ m $\cdot$ s. The first integral of energy of the torque-free gyrostat is $H_0=100$ N $\cdot$ m. The first integral of areas of the torque-free gyrostat is $G_p=47.1845$ N $\cdot$ m $\cdot$ s. These physical parameters are acquired through using rigorous algorithms that can be found in Ref. 16. The homoclinic orbits with respect to the angular velocities of the torque-free gyrostat are given in Eqs. (45–47). From Eq. (48), one can determine whether or not the stable and unstable manifolds of the periodic orbits intersect. Because the integrand in Eq. (48) is the Poisson bracket $\{H_0, H_1/\Omega\}$ with respect to the variables l and L, different perturbation potential functions give different PAM integrals.

Case 1

Case 1 is that of a heavy gyrostat (rotating about a fixed point) under the action of the gravity torques. From Eqs. (89–91) associated with Eqs. (54–87), one can derive the following partial derivatives:

$$I_0 = \frac{\partial I_0}{\partial(\cos I)}\cos I \tag{184}$$

$$I_c = \frac{\partial I_c}{\partial(\sin I)}\sin I \tag{185}$$

$$I_s = \frac{\partial I_s}{\partial(\sin I)}\sin I \tag{186}$$

Only multiplication and addition, both numerically and symbolically, are involved because I_0, I_c, and I_s are linear functions of $\sin I$ and $\cos I$. Substituting Eqs. (184–186) into Eq. (88), one obtains

$$M(g_0) = \cos I\frac{\partial I_0}{\partial(\cos I)} + \cos(g_0)\sin I\frac{\partial I_c}{\partial(\sin I)} + \sin(g_0)\sin I\frac{\partial I_s}{\partial(\sin I)} \tag{187}$$

From Eq. (187) by setting $M(g_0)=0$, one observes that the condition for the PAM function to have simple zeros with respect to the angle parameter g_0 becomes

$$\cot I = -\left[\cos(g_0)\frac{\partial I_c}{\partial(\sin I)} + \sin(g_0)\frac{\partial I_s}{\partial(\sin I)}\right]\Big/\frac{\partial I_0}{\partial(\cos I)} \tag{188}$$

Given I_{xx}, I_{yy}, I_{zz}, h_x, h_y, h_z, G_p, H_0, x_0, y_0, and z_0, the relationship between the Deprit variables[11] g_0 and I of the heavy gyrostat under the action of the gravity torques is established in Eq. (188). The constraint equation (188) between the parameter I and g_0 must be satisfied for chaotic motions to occur. In the simulation of the chaotic attitude dynamics, one makes use of the fourth-order Runge–Kutta algorithms (see Ref. 31) as a numerical integration method and use of quaternions as the attitude representation.[1] The advantage of using quaternions is that one will not encounter the singularity problems when dealing with the numerical calculations of the attitude dynamic equations (1–6) or the Hamiltonian equations (21–26) in terms of the Deprit variables.[11] In addition, by numerically integrating the dynamic equations and the kinetic differential equations in terms of quaternions, one can draw the Poincaré section to check. The kinetic differential equations describing the attitude of the heavy gyrostat or the gyrostat satellite in terms of Euler parameters (quaternions) are given as follows:

$$\frac{dq_x}{dt} = \frac{1}{2}(\omega_x q_0 - \omega_y q_z + \omega_z q_y - nq_0) \tag{189}$$

$$\frac{dq_y}{dt} = \frac{1}{2}(\omega_x q_z + \omega_y q_0 - \omega_z q_x + nq_z) \tag{190}$$

$$\frac{dq_z}{dt} = \frac{1}{2}(-\omega_x q_y + \omega_y q_x + \omega_z q_0 - nq_y) \tag{191}$$

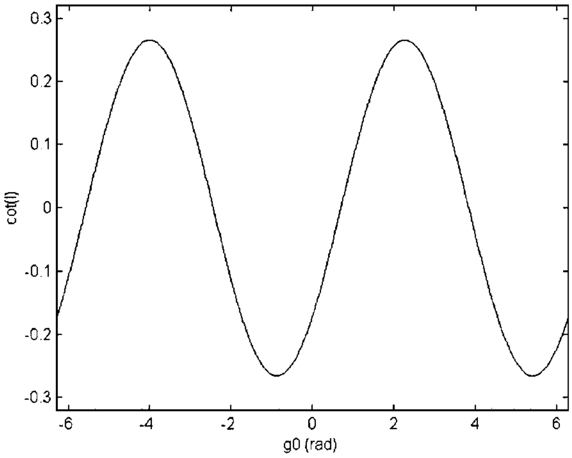
$$\frac{dq_0}{dt} = \frac{1}{2}(-\omega_x q_x - \omega_y q_y - \omega_z q_z + nq_x) \tag{192}$$

$$q_x^2 + q_y^2 + q_z^2 + q_0^2 = 1 \tag{193}$$

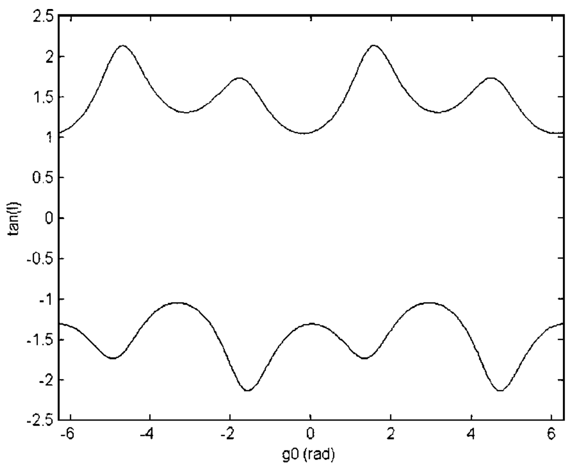
where $n=0$ refers to heavy gyrostats under the action of gravity torques. The relation between the direction cosines and quaternions is expressed as follows:

$$\gamma_x = 1 - 2\left(q_y^2 + q_z^2\right) \tag{194}$$

$$\gamma_y = 2(q_x q_y - q_z q_0) \tag{195}$$



a)



b)

Fig. 4 Relation graphs between the Deprit's variables[11] I and g_0 of the gyrostat under the action of a) gravity torques and b) gradient-gravity torques.

$$\gamma_z = 2(q_x q_z + q_y q_0) \tag{196}$$

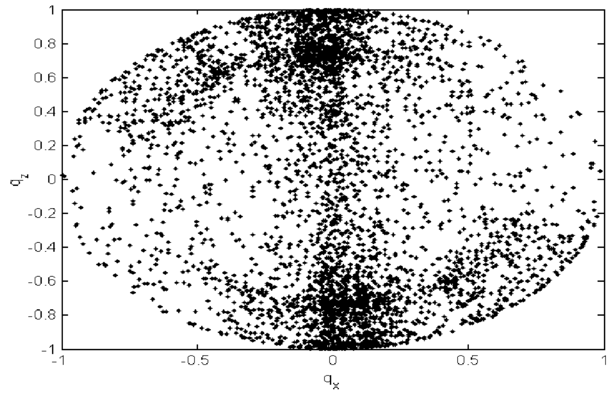
Figure 4a depicting Eq. (188) shows that the PAM function in Eq. (88) or (96) has simple zeros. According to the criteria established in Secs. III and IV, one can conclude that the attitude motions of the corresponding gyrostat under the action of the gravity torques are chaotic. This is numerically demonstrated by using the Poincaré sections in Fig. 5a. In the numerical simulation, the position parameters of the mass center and the weight of the heavy gyrostat

$$x_0 = 0.1000\ \text{m}, \qquad y_0 = 0.0997\ \text{m}$$
$$z_0 = 0.2779\ \text{m}, \qquad M_g = 100\ \text{N} \tag{197}$$

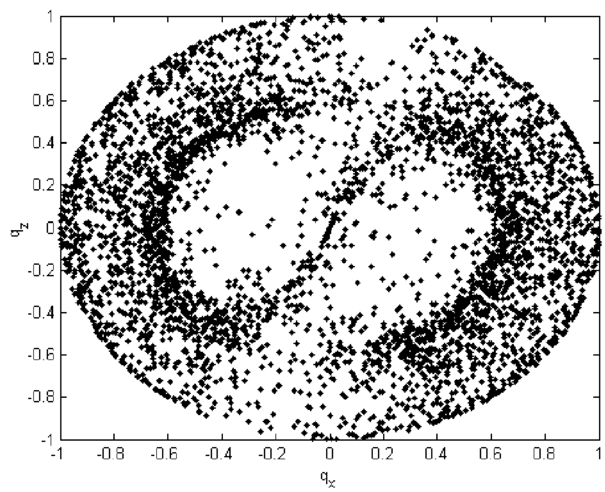
are assumed.

Case 2

Case 2 is that of a gyrostat satellite under the action of the gravity-gradient torques. Here one deals with a concrete example of the chaotic attitude motion of the gyrostat satellite under the action of the gravity-gradient torques. The differential equations used for simulation consist of Eqs. (1–3) associated with Eqs. (129–131) and (189–192) with the constraint equation (193), taking $n = \sqrt{(\mu/\rho_c^3)} \neq 0$, $\mu = 3.986005 \times 10^{14}\ \text{m}^3/\text{s}^2$ and $\rho_c = 7{,}178{,}137$ m. The PAM integral in Eq. (165) has simple zeros if Eq. (182) holds. Figure 4b depicts Eq. (182). For the gyrostat satellite, the four roots of Eq. (171) associated with Eq. (165) are as follows:



a)



b)

Fig. 5 Ponicaré map ($q_x - q_z$) at the plane ($q_y = 0$), 4146 points: a) gyrostat under the action of gravity torques and b) gyrostat satellite under the action of gravity-gradient torques.

$$\tan(g_{01}) = -1.1741, \qquad \tan(g_{02}) = -1.3097$$
$$\tan(g_{03}) = 0.5974, \qquad \tan(g_{04}) = 1.8904 \tag{198}$$

The existence of these roots in Eq. (171) means that the corresponding attitude motions of the gyrostat satellite under the action of the gravity-gradient torques are chaotic. This is numerically demonstrated by using the Poincaré section method in Fig. 5b. By computation based on the subroutines of the MATLAB software (Version 6.0.0.88, Release 12), all of the roots of Eq. (105) are complex, such as

$$0.0673 \pm 1.0731i, \qquad -0.0445 \pm 1.0721i$$
$$-0.0636 \pm 1.0501i, \qquad 0.0368 \pm 1.0457i$$
$$-0.0501 \pm 1.0014i, \qquad -0.0456 \pm 0.9952i$$
$$0.0524 \pm 1.0026i, \qquad 0.0480 \pm 0.9906i$$
$$0.0302 \pm 0.9985i, \qquad 0.0151 \pm 0.9769i$$
$$-0.0140 \pm 0.9742i, \qquad -0.0300 \pm 0.9329i$$

where $i = \sqrt{-1}$ is the imaginary unit. Similarly, the eight roots of the polynomial Eq. (114) are also complex, that is,

$$-0.0005 \pm 1.0415i, \qquad 0.0212 \pm 1.0321i$$
$$-0.0023 \pm 0.9826i, \qquad -0.0041 \pm 0.9737i$$

Therefore, there is no singularity in $z \in (-1, +1)$ in the PAM integrals when the gyrostat is under the action of the conservative forces discussed earlier.

VII. Conclusions

The chaotic behavior of the gyrostat under the action of conservative forces was investigated. These were achieved by applying the PAM integral, which was developed by Holmes and Marsden,[6] for the attitude model equations. These equations were transformed from the Euler dynamic equations in terms of the angular velocities and Euler angles to the Hamiltonian equations in terms of the canonical Deprit variables[11] (also see Refs. 15–17). Criteria for the occurrence of chaos of the attitude motions of the gyrostat under the action of either the gravity torques or the gravity-gradient torques were derived. The methodologies presented are suitable for the exploration of a wide range of chaotic attitude dynamics problems under the assumption that the external disturbance makes the PAM function integral have simple zeros. The PAM function equation establishes the relationship between the Deprit variables[11] of the torque-free gyrostat and the physical parameters of the disturbed gyrostat. With the use of the fourth-order Runge–Kutta integration algorithms, the long-term behaviors in terms of the Euler parameters (quaternions) were simulated. The Poincaré sections from the phase portrait curve showed that the attitude chaotic motions of the gyrostat disturbed by either the gravity torques or the gravity-gradient torques were similar to random motions or bounded nonperiodic motions. The numerical simulation shows that the chaotic dynamics is sensitive to the initial conditions. The PAM theory can be used to predict the onset of chaos (in the sense of Smale's horseshoe) in a gyrostat perturbed by rather general gravity torques or gravity-gradient torques under the following presuppositions: 1) The dynamic system can be described by the Hamiltonian equations. 2) Homoclinic solutions to the torque-free gyrostat exist. 3) No singularities in the PAM integral exist. 4) The disturbing torques are small. The results also provide a theoretical foundation for numerical analyses on the attitude chaotic dynamics of the single rigid gyroscope or liquid-filled top. This can be a mechanical model of planets with a liquid core according to the Rumyantsev equivalent principle between the liquid-filled solid body and the gyrostat under some appropriate assumptions.[26]

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